A Note on Grid Drawings of Plane Graphs

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November 24, 1998

Technical Report UCR-CS-98-07
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Abstract

We consider the problem of drawing plane graphs such that the vertices are represented by grid points and edges are drawn as straight lines. It is an open problem to determine the minimum size $\mu \times \lambda$ of grids that admit straight-line drawings of all n-vertex plane graphs. Algorithms are known for producing drawings in grids of size $(n-2) \times (n-2)$ and $\lfloor 2(n-1)/3 \rfloor \times \lfloor 8(n-1)/3 \rfloor$. The currently known lower bound is that each dimension satisfies $\mu, \lambda \geq \lfloor 2(n-1)/3 \rfloor$. In this note, we slightly improve this lower bound by showing that $\mu = \lfloor 2(n-1)/3 \rfloor$ implies $\lambda \geq \lceil 2n/3 \rceil$. Thus it is impossible to achieve the minimum width and height simultaneously.

1 Introduction

In this paper we deal with the following problem: We are given a plane graph G, that is a planar graph with a specified embedding. We want to map the vertices of G into integer grid points in such a way that the edges between them can be drawn as straight, non-intersecting line segments, and the resulting drawing is consistent with the planar embedding of G. We call such mappings *grid drawings*.

In 1988 de Fraisseix, Pach and Pollack [FPP88, FPP90] proved that each plane graph with n vertices can be embedded into a $(2n-4) \times (n-2)$ grid, and Chrobak and Payne [CP95] gave a simple, linear-time implementation of their technique. (Throughout the paper we assume that $n \geq 3$.) Schnyder [Sc90] presented a different method, based on barycentric representations, that

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produces grid drawings of size $(n-2) \times (n-2)$. Kant [Ka93] proved that each 3-connected plane graph has a convex drawing in a $(2n-4) \times (n-2)$ grid, and the grid size was later improved to $(n-2) \times (n-2)$ by Schnyder and Trotter [ST92] and, independently, by Chrobak and Kant [CK93]. In [CN98], Chrobak and Nakano gave an algorithm that produces straight-line drawings in a $\lfloor 2(n-1)/3 \rfloor \times \lfloor 8(n-1)/3 \rfloor$ grid. All these algorithms can be implemented in linear time.

We say that a $\mu \times \lambda$ grid is *n-universal* if each plane graph can be embedded into it with straight lines. The obvious question is: what is the minimum size of an n-universal grid? The known lower bound [FPP88, CN98] is that each dimension satisfies $\mu, \lambda \geq \lfloor 2(n-1)/3 \rfloor$. By [CN98], the $\lfloor 2(n-1)/3 \rfloor$ lower bound for each dimension is tight, even if the other one is unrestricted.

In this paper we slightly improve the lower bound on the grid size by proving the following theorem:

Theorem 1 *Let $n \geq 6$, and suppose that the $\mu \times \lambda$ grid is n-universal, where $\mu \leq \lambda$. Then $\mu \geq \left\lfloor \frac{2(n-1)}{3} \right\rfloor$, and $\mu = \left\lfloor \frac{2(n-1)}{3} \right\rfloor$ implies $\lambda \geq \left\lceil \frac{2n}{3} \right\rceil$.*

In particular, we obtain that it is impossible to simultaneously achieve the minimum width and height. The rest of the paper is devoted to the proof of Theorem 1.

2 Proof of Theorem 1

Let $w(n) = \lfloor 2(n-1)/3 \rfloor$. Given a straight-line grid drawing E of a plane graph, by *width*(E) and *height*(E) we denote the width and height of E. For a given plane graph G with n vertices, define

$$H(G) \quad = \quad \min_{width(E) \leq w(n)} height(E)$$

where the minimum is over all drawings E of G of width at most $w(n)$. Define also:

$$h(n) \quad = \quad \max_{|G|=n} H(G)$$

where the minimum is over all plane graphs G with n vertices. Of course, we can assume that G is maximal (triangulated). We know that $h(n) \geq \lfloor 2(n-1)/3 \rfloor$ (by [FPP88, CN98]).

Lemma 1 (a) *For $n = 3, \ldots, 8$ there is a graph J_n for which $H(J_n) \geq n-2$ for $n = 3, \ldots, 8$.*

(b) *For $n = 3, \ldots, 8$ the function $h(n)$ is as in the table below:*

n	3	4	5	6	7	8
$w(n)$	1	2	2	3	4	4
$h(n)$	1	2	3	4	5	≥ 6

We will prove the lemma in the following sections. We show now how Lemma 1 implies Theorem 1.

Proof of Theorem 1: The proof that $\mu \geq w(n)$ was given in [FPP88, CN98]. For completeness, the proof goes as follows: Let $I_n = G_1^n$, $n = 3, 4, 5$, where the graphs G_1^n are given in Figure 1. It is easy to see, by inspection, that each G_1^n requires width $w(n)$.

Denote by $\nabla(G)$ the graph obtained from G by drawing a triangle around it and connecting its vertices to the exterior vertices of G in some arbitrary way. Let $I_n = \nabla(I_{n-3})$, for $n \geq 6$. Then I_n must use two more x-coordinates than I_{n-3}. By simple induction, I_n requires width $w(n)$ for any $n \geq 3$.

Now we show that $\mu = w(n)$ implies $\lambda \geq \lceil 2n/3 \rceil$. Let J_n be the graphs from Lemma 1(a). For $n = 6, 7, 8$, $\lceil 2n/3 \rceil = n - 2$, so each J_6, J_7 and J_8 satisfies $H(J_n) \geq \lceil 2n/3 \rceil$. Again, for $n \geq 9$, let $J_n = \nabla(J_{n-3})$. Then each J_n requires two more y-coordinates than J_{n-3}, and, by induction, the proof is complete. □

3 Some Useful Facts

External degree sequences. If the external vertices of G are p, q, r with degrees $\deg(p) = a$, $\deg(q) = b$, $\deg(r) = c$ and $a \geq b \geq c$, then the sequence (a, b, c) is called the *external degree sequence.*

Lemma 2 *Let $n \geq 6$. Define $G' = G - \{p, q, r\}$. Then G' is a plane graph with the following properties: (i) G' has $3n-a-b-c-3$ edges, (ii) G' is connected, and each 2-connected component of G' is internally triangulated.*

Proof : G has $3n - 6$ edges. If we subtract 3 exterior edges and $a + b + c - 6$ edges from the exterior to the interior vertices, we get $3n - a - b - c - 3$. This proves (i). Part (ii) is obvious. □

Lemma 3 *Let $n \geq 6$. The external degree sequence has the following properties: (i) $a \leq n - 1$, (ii) $c \geq 3$ and $b \geq 4$, (iii) $12 \leq a + b + c \leq 2n + 1$.*

Proof : Parts (i) and (ii) are easy. To prove (iii), let $G' = G - \{p, q, r\}$. Then G' has $3n - a - b - c - 3$ edges, by Lemma 2. On the other hand, G' is connected and has $n - 3$ vertices, so it has between $n-4$ and $3(n-3)-6 = 3n-15$ edges. Therefore $n-4 \leq 3n-a-b-c-3 \leq 3n-15$, which implies (iii). □

Classification of graphs with an exterior degree 3. The following fact is obvious, and it will help to classify graphs that have one exterior vertex of degree 3.

Fact 1 *Let G be a maximal plane graph with $n \geq 5$ vertices. Let p, q, r be the external vertices of G with degrees $a \geq b \geq c$, where $c = 3$. Then $G' = G - \{r\}$ is a maximal plane graph with $n - 1$ vertices, and in G' vertices p and q have degrees $a - 1$ and $b - 1$, respectively.*

Here is how we can classify n-vertex graphs that have one exterior vertex of degree 3: For each external degree sequence $(a, b, 3)$, list all $(n-1)$-vertex graphs G' that have external vertices, say u, v, of degrees $a-1$ and $b-1$, and add a new external vertex adjacent to all external vertices of G' such that u, v remain external. Remove all redundant graphs (isomorphic copies).

Classification of graphs with an exterior degree 4. The following fact is obvious, and it will help to classify graphs that have one exterior vertex of degree 4 (and no external degree 3). Define a *4-maximal* plane graph to be an internally triangulated, 2-connected plane graph with 4 vertices on the exterior face.

Fact 2 *Let G be a maximal plane graph with $n \geq 6$ vertices. Let p, q, r be the external vertices of G with degrees $a \geq b \geq c$, where $c = 4$. Then $G' = G - \{r\}$ is a 4-maximal plane graph with $n-1$ vertices, and in G' vertices p and q have degrees $a-1$ and $b-1$, respectively.*

This also yields a way to classify graphs with an external degree 4. One problem is that we don't have a classification of plane graphs which are internally triangulated and have 4 external vertices. But each such graph can be obtained from a maximal plane graph by removing an edge.

4 Proof of Lemma 1 for $n = 4, 5$

For $n = 3, 4, 5$ we only have one maximal plane graph, as below:

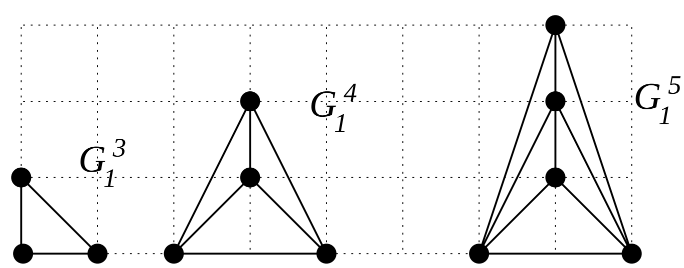


Figure 1: Maximal plane graphs with $n = 3, 4, 5$.

For $n = 3, 4$, it is trivial that G_1^3 and G_1^4 are the only graph. For $n = 5$, we have two interior vertices that must be connected by an edge, and there is only one way to add other edges (up to isomorphism), so we get G_1^5.

In Part (a) of Lemma 1, we take $J_n = G_1^n$ for $n = 3, 4, 5$. That $h(n) \geq 2$ for $n = 4, 5$ is trivial, and Figure 1 shows the drawing of G_1^4 in a 2×2 grid, and G_1^5 in a 2×3 grid. So $h(4) = 2$. To show that $h(5) = 3$, it remains to show that G_1^5 cannot be drawn in a 2×2 grid. But this follows from the observation that the 2×2 grid has only one interior point, while G_1^5 needs two interior points.

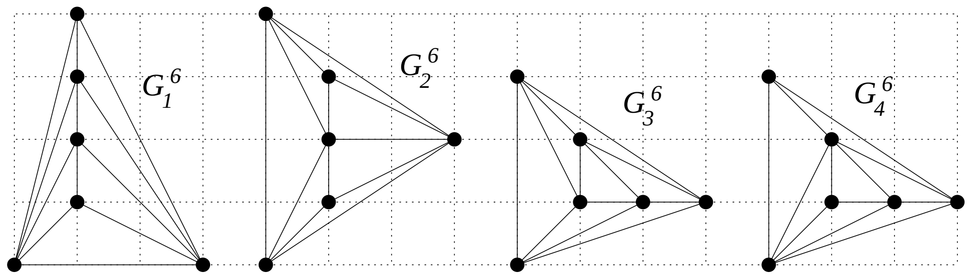


Figure 2: Graphs with 6 vertices.

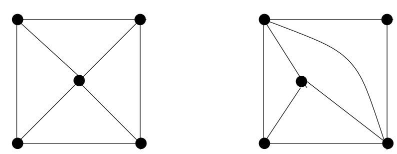
5 Proof of Lemma 1 for $n = 6$

By Lemma 3(iii) we have $12 \le a+b+c \le 13$. Using also (i) and (ii), the following are all possible external degree sequences: 553, 544, 543, 444.

Claim: There are four maximal plane graphs with 6 vertices, as in Figure 2.

Proof: We show that each sequences gives just one graph. Consider first 553 and 543, that have a vertex of degree 3. By Fact 1 these graphs must be obtained from G_1^5 by adding an external vertex of degree 3, and there are just two ways to do so, getting G_1^6 and G_4^6.

We show that sequences 444, 544 give just one graph each, using Fact 2. There are two 4-maximal plane graphs with 5 vertices:

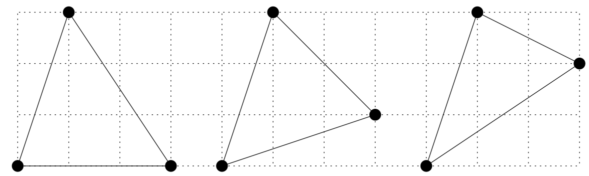


From the first one, by adding an external vertex of degree 4, we get G_3^6, because of symmetry. In the second graph, we need to cover the degree-2 vertex, and there is just one way to do it up to symmetry, giving G_2^6.

Claim H6: $H(G_2^6) = 4$ and $h(6) = 4$.

Taking $J_6 = G_2^6$, Claim H6 will imply Lemma 1 for $n = 6$.

Proof of Claim H6: Figure 2 shows that $h(6) \le 4$. It is sufficient to prove that $H(G_2^6) \ge 4$. Towards contradiction, suppose that G_2^6 is embedded into a 3×3 grid. Without loss of generality, one vertex is in the corner of the grid, say at $(0, 0)$. (Otherwise we would have to use a smaller grid, 3×2, which only has two internal points.) Then there are only three external triangles (up to symmetry) with at least three internal points:



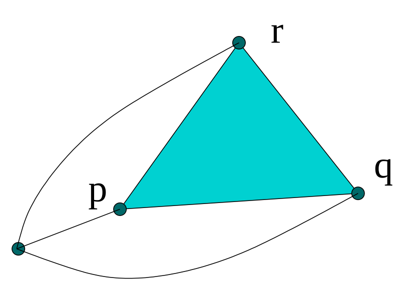
An interior point of an external triangle is called (0, 1, 1)-*central* if, after connecting it to the exterior vertices, two of the resulting triangles have an interior point. Clearly, in any drawing of G_2^6 the exterior triangle must have a (0, 1, 1)-central point. However, none of the triangles above contain a (0, 1, 1)-central point. So it is not possible to draw G_2^6 in a 3×3 grid.

6 Proof of Lemma 1 for $n = 7$

By Lemma 3, the following are all possible external degree sequences: 663, 654, 653, 644, 643, 555, 554, 553, 544, 543, 444.

Claim: Figure 3 lists all non-isomorphic plane graphs with 7 vertices.

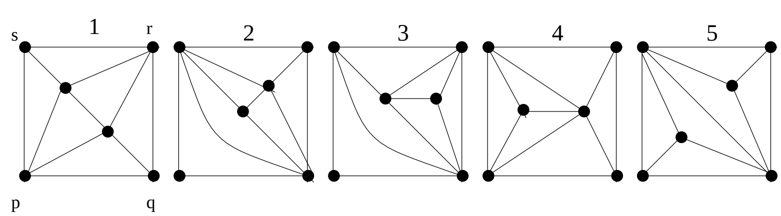
Proof: We will first analyze all graphs with an external degree 3. These are obtained from the G_i^6 by adding an external vertex of degree 3. Let p, q, r denote the lower left, right and top vertices in Fig 2. By $p(G_i^6)$ we denote the graph obtained from G_i^6 by adding a degree-3 vertex that covers p, as below:



Similarly we define $q(G_i^6)$ and $r(G_i^6)$. Then, ignoring symmetries, we get

$$p(G_1^6) = G_4^7 \quad r(G_1^6) = G_{10}^7 \quad p(G_2^6) = G_{16}^7 \quad q(G_2^6) = G_7^7$$
$$p(G_3^6) = G_5^7 \quad p(G_4^6) = G_1^7 \quad q(G_4^6) = G_3^7 \quad r(G_4^6) = G_6^7$$

Now we analyze graphs with an external degree 4. All 4-maximal plane graphs with 6 vertices are listed below.



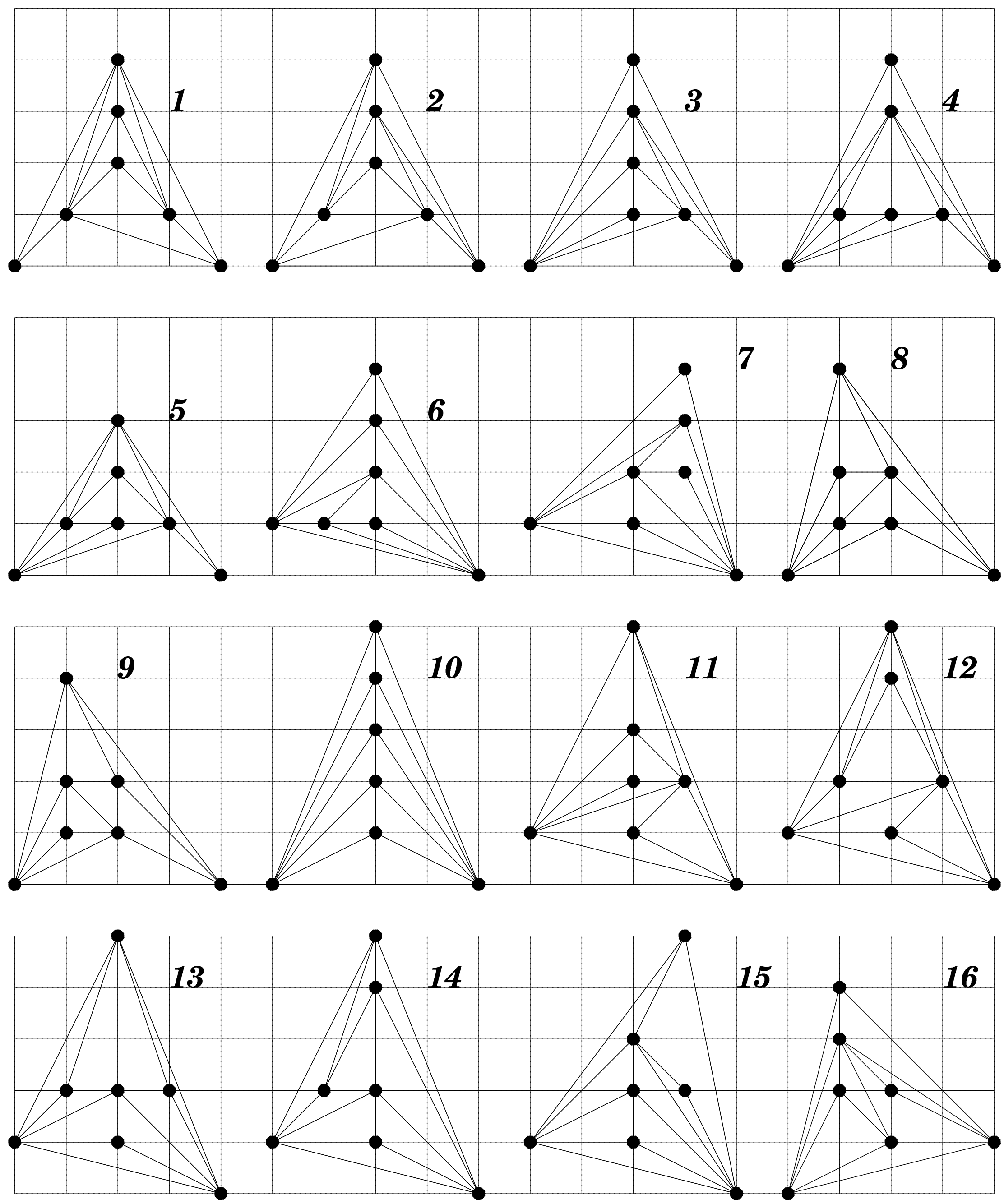


Figure 3: Maximal plane graphs with 7 vertices.

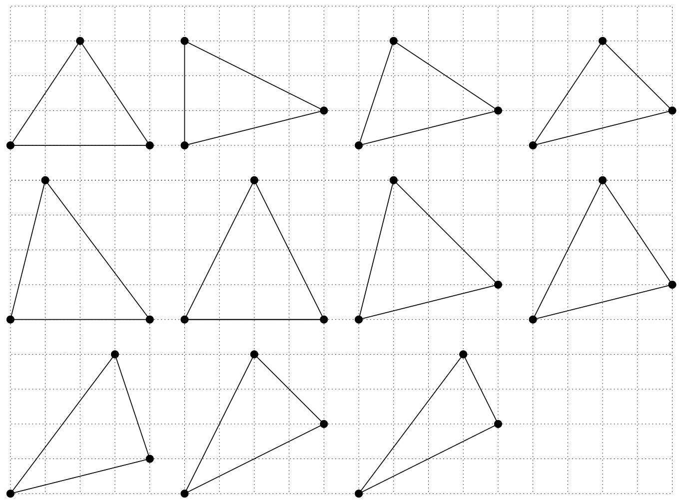


Figure 4: External 4×5 triangles with 5 interior points.

We name these graphs H_i, $i = 1, 2, 3, 4, 5$. To see that this is a complete list, note that each such graph must have a vertex of degree 2 or 3. If it has a vertex of degree 2, it can be obtained from G_1^5 by adding such a vertex. This gives us graphs H_2 and H_3. Otherwise, it has a vertex of degree 3, and it was obtained from one of 4-maximal plane graphs by adding a vertex of degree 3. This gives us H_1, H_4 and H_5.

By $pq(H_i)$ we denote the graph obtained from H_i by adding a degree-4 vertex that covers p and q. Similarly we define $qr(H_i)$, $rs(H_i)$ and $sp(H_i)$. Omitting symmetries and graphs with an external degree 3, we generate the following graphs:

$$\begin{array}{llll} pq(H_1) = G_8^7 & pq(H_2) = G_{11}^7 & pq(H_3) = G_{12}^7 & sp(H_3) = G_{15}^7 \\ pq(H_4) = G_9^7 & qr(H_4) = G_{14}^7 & sp(H_4) = G_2^7 & pq(H_5) = G_{11}^7 \end{array}$$

The only external degree sequence we did not consider is $5, 5, 5$. Each pair of external vertices u, v shares a common interior neighbor n_{uv}. The n_{uv} must be different for all u, v, for otherwise the graph would have an exterior vertex of degree 3. There is one more vertex in the interior and it must be adjacent to all exterior vertices. This gives G_{13}^7.

Claim H7: $H(G_{13}^7) = 5$ and $h(7) = 5$.

Taking $J_6 = G_{13}^7$, Claim H7 will imply Lemma 1 for $n = 7$.

Proof of Claim H7: Figure 3 shows drawings of all graphs G_i^7 of width 4 and height at most 5. So $h(7) \leq 5$. It remains to show that $H(G_i^7) = 5$ for some i.

We prove that $H(G_{13}^7) \geq 5$. For suppose that G_{13}^7 is drawn in a 4×4 grid. There are 11 ways (up to symmetry) to draw the external triangle in a 4×4 grid, in such a way that it contains four grid points in the interior, see Figure 4.

Given an embedding of the exterior triangle, define a grid point t to be $(1, 1, 1)$-*central* if t is interior and if for each pair (u, v) of exterior vertices the triangle (t, u, v) has at least one interior point. An embedding of an exterior triangle can be extended to an embedding of G_{13}^7 only if it contains a $(1, 1, 1)$-central point. But no triangle in Figure 4 contains such a point. Therefore G_{13}^7 cannot be drawn in a 4×4 grid. We conclude that $h(7) = 5$.

7 Proof of Lemma 1 for $n = 8$

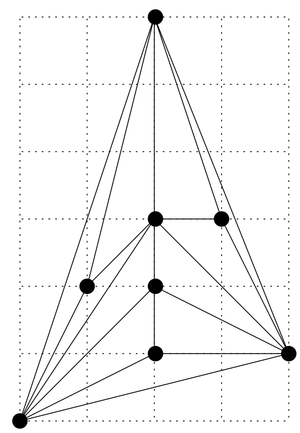


Figure 5: Graph G_{13}^8.

In this section we will prove that $h(8) \geq 6$. Consider graph G_{13}^8 obtained from G_{13}^7 by adding another interior vertex adjacent to two exterior vertices, see Figure 5. Taking $J_8 = G_{13}^8$, Claim H8 below will imply Lemma 1 for $n = 8$.

Claim H8: $H(G_{13}^8) \geq 6$.

Proof of Claim H8: First, $H(G_{13}^8) \geq 5$, because $H(G_{13}^7) \geq 5$ and G_{13}^7 is a subgraph of G_{13}^8. Thus if we could draw $H(G_{13}^7)$ in a 4×5 grid, it would have to have height exactly 5. Now we consider the external triangle. By symmetry and by the argument above, we can assume that one vertex is at $(0, 0)$, another is at $x = 4$, and the third is at $y = 5$.

Given an exterior triangle and its interior point t, we say that t is $(1, 1, 2)$-*central* if among the three triangles obtained by drawing lines from t to the exterior vertices, one has at least two interior points and two have at least one interior point each.

Figure 6 contains a complete list of the exterior triangles of size 4×5 that have at least five grid points in the interior. An embedding of the exterior triangle can be extended to an embedding of G_{13}^8 only if it contains a $(1, 1, 2)$-central point. Therefore only triangles 5, 11, 13 may be extensible to an embedding of G_{13}^8. But each of these three triangles has the following property: if t is the $(1, 1, 2)$-central point and u, v are external points for which the triangle (t, u, v) has at least two interior points, then all these interior points are colinear with either u or v, and thus not two of them can be connected to both u and v.

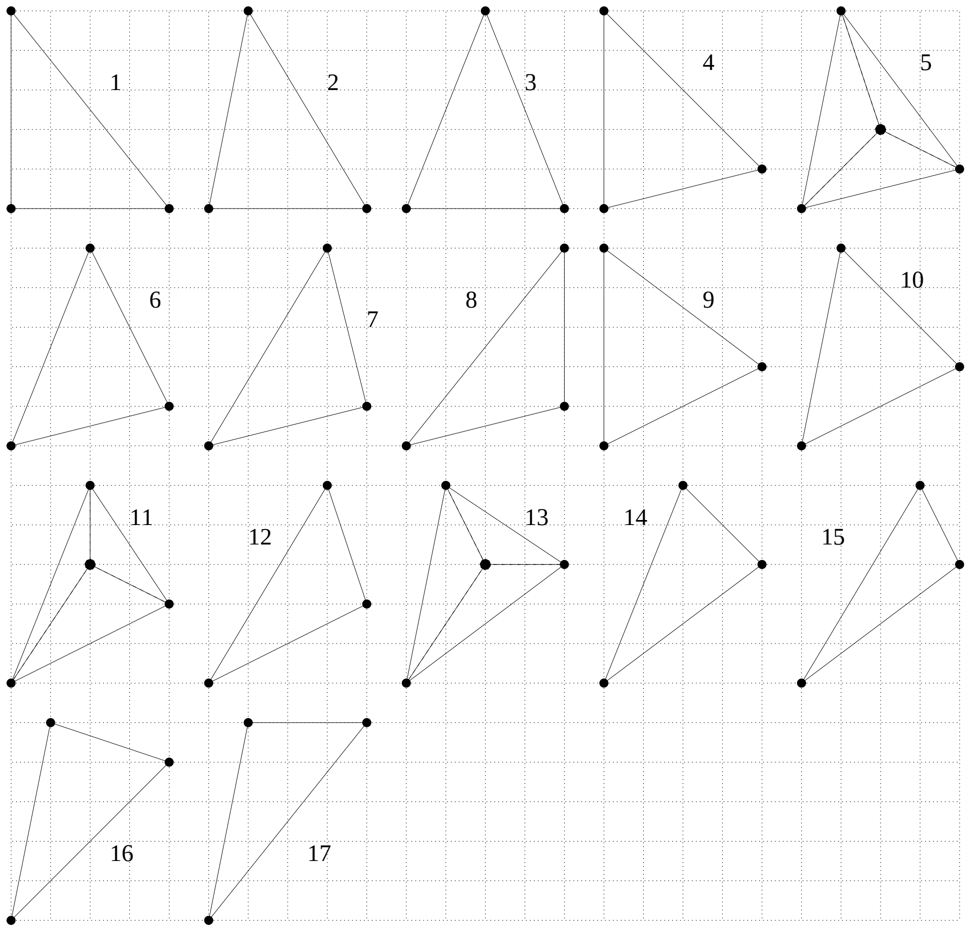


Figure 6: External 4×5 triangles with at least 5 interior points.

Therefore G^8_{13} cannot be drawn in a 4×5 grid, and we conclude that $H(G^8_{13}) = 6$.

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